

# Entropy of an extremal regular black hole

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## Abstract

We introduce a magnetically charged extremal regular black hole in the coupled system of Einstein gravity and nonlinear electrodynamics. Its near horizon geometry is given by  $AdS_2 \times S^2$ . It turns out that the entropy function approach does not automatically lead to a correct entropy of the Bekenstein-Hawking entropy. This contrasts to the case of the extremal Reissner-Norström black hole in the Einstein-Maxwell theory. We conclude that the entropy function approach does not work for a magnetically charged extremal regular black hole without singularity, because of the nonlinearity of the entropy function.

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# 1 Introduction

Growing interest in the extremal black holes is motivated by their unusual and not fully understood nature. The problems of entropy, semiclassical configurations, interactions with matter, or information paradox have not been resolved yet. Moreover, the near-horizon region is also of interest apart from their whole structure and behavior. Indeed, one can generate new exact solutions [1] applying appropriate limiting procedure in the geometry of the extremal and near extremal black holes. In particular, of the equal importance is the question of the nature of singularities that reside in the centers of most black holes hidden to an external observer. Regular black holes (RBHs) have been considered, dating back to Bardeen [2], for avoiding the curvature singularity beyond the event horizon [3]. Their causal structures are similar to the Reissner-Nordström black hole with the singularity replaced by de Sitter space-time [4]. Hayward has discussed the formation and evaporation process of a RBH [5, 6]. A more rigorous treatment of the evaporation process was carried out for the renormalization group improved black hole [7]. The noncommutativity also provides another RBH: noncommutative black hole [8].

Among various RBHs known to date, especially intriguing black holes are from the known action of Einstein gravity and nonlinear electrodynamics. The solutions to the coupled equations were found by Ayón-Beato and García [9] and by Bronnikov [10]. The latter describes a magnetically charged black hole, and provides an interesting example of the system that could be both regular and extremal. Also its simplicity allows exact treatment such that the location of the horizons can be expressed in terms of the Lambert functions [11]. Moreover, Matyjasek investigated the magnetically charged extremal RBH with the near horizon geometry of  $AdS_2 \times S^2$  and its relation with the exact solutions of the Einstein field equations [12, 13]. Only this type of RBHs can be employed to test whether the entropy function approach is or not suitable for obtaining the entropy of the extremal RBHs.

On the other hand, string theory suggests that higher curvature terms can be added to the Einstein gravity [14]. Black holes in higher-curvature gravity [15] were extensively studied during two past decades culminating in recent spectacular progress in the microscopic string calculations of the black hole entropy. For a review, see [16]. In theories with higher curvature corrections, classical entropy deviates from the Bekenstein-Hawking value and can be calculated using Wald's formalism [17]. Remarkably, it still exhibits exact agreement with string theory quantum predictions at the corresponding level, both in the BPS [18, 19] and non-BPS [20] cases. In some supersymmetric models with higher curvature terms, exact classical solutions for static black holes were obtained [19]. Recently,

Sen has proposed a so-called “entropy function” method for calculating the entropy of  $n$ -dimensional extremal singular black holes, which is effective even for the presence of higher curvature terms. Here the extremal black holes are characterized by the near horizon geometry  $AdS_2 \times S^{n-2}$  and corresponding isometry [22]. It states that the entropy of such kind of extremal black holes can be obtained by extremizing the “entropy function” with respect to some moduli on the horizon. This method does not depend on supersymmetry and has been applied to many solutions in supergravity theory. These are extremal black holes in higher dimensions, rotating black holes and various non-supersymmetric black holes [23, 24].

In this paper we consider a magnetically charged RBH with near horizon geometry  $AdS_2 \times S^2$  in the coupled system of the Einstein gravity and nonlinear electrodynamics [12, 13]. The solution is parameterized by two integration constants and a free parameter. Using the boundary condition at infinity, the integration constants are related to Arnowitt-Deser-Misner (ADM) mass  $M$  and magnetic charge  $Q$ , while the free parameter  $a$  is adjusted to make the resultant line element regular at the center. Here we put special emphasis on its extremal configuration because it has the same near horizon geometry  $AdS_2 \times S^2$  of the extremal Reissner-Nordström black hole ( $a = 0$  limit), but it is regular inside the event horizon. In this work, we investigate whether the entropy function approach does work for deriving the entropy of a magnetically charged extremal regular black hole without singularity.

As a result, we show that the entropy function approach proposed by Sen does not lead to a correct form of the Bekenstein-Hawking entropy of an extremal RBH. However, using the generalized entropy formula based on Wald’s Noether charge formalism [25], we find the correct entropy.

## 2 Magnetically charged RBH

We briefly recapitulate a magnetically charged extremal RBH with the special emphasis put on the near horizon geometry  $AdS_2 \times S^2$  and its relation with the exact solutions of the Einstein equations [12, 13]. Let us begin with the following action describing the Einstein gravity-nonlinear electrodynamics

$$S = \int d^4x \sqrt{-g} \mathcal{L} = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - \mathcal{L}(B)]. \quad (1)$$

Here  $\mathcal{L}(B)$  is a functional of  $B = F_{\mu\nu}F^{\mu\nu}$  defined by

$$\mathcal{L}(B) = B \cosh^{-2} \left[ a \left( \frac{B}{2} \right)^{1/4} \right], \quad (2)$$

where the free parameter  $a$  will be adjusted to guarantee regularity at the center. In the limit of  $a \rightarrow 0$ , we recover the Einstein-Maxwell theory in favor of the Reissner-Nordström black hole. First, the tensor field  $F_{\mu\nu}$  satisfies equations

$$\nabla_\mu \left( \frac{d\mathcal{L}(B)}{dB} F^{\mu\nu} \right) = 0, \quad (3)$$

$$\nabla_\mu {}^* F^{\mu\nu} = 0, \quad (4)$$

where the asterisk denotes the Hodge duality. Then, differentiating the action  $S$  with respect to the metric tensor  $g_{\mu\nu}$  leads to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \quad (5)$$

with the stress-energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi} \left( \frac{d\mathcal{L}(B)}{dB} F_{\rho\mu} F_\nu^\rho - \frac{1}{4} g_{\mu\nu} \mathcal{L}(B) \right). \quad (6)$$

Considering a static and spherically symmetric configuration, the metric can be described by the line element

$$ds^2 = -G(r)dt^2 + \frac{1}{G(r)}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (7)$$

with the metric function

$$G(r) = 1 - \frac{2m(r)}{r}. \quad (8)$$

Here,  $m(r)$  is the mass distribution function. Solving the full Einstein equation (5) leads to the mass distribution

$$m(r) = \frac{1}{4} \int^r \mathcal{L}[B(r')] r'^2 dr' + C, \quad (9)$$

where  $C$  is an integration constant. In order to determine  $m(r)$ , we choose the purely magnetic configuration as follows

$$F_{\theta\phi} = Q \sin \theta \rightarrow B = \frac{2Q^2}{r^4}. \quad (10)$$

Hereafter we assume that  $Q > 0$  for simplicity. Considering the condition for the ADM mass at infinity ( $m(\infty) = M = C$ ), the mass function takes the form

$$m(r) = M - \frac{Q^{3/2}}{2a} \tanh\left(\frac{aQ^{1/2}}{r}\right). \quad (11)$$

Finally, setting  $a = Q^{3/2}/2M$  determines the metric function completely as

$$G(r) = 1 - \frac{2M}{r} \left(1 - \tanh\frac{Q^2}{2Mr}\right). \quad (12)$$

At this stage we note that the form of metric function  $G(r)$  is obtained when using the mass distribution (9) and boundary condition. However, we will show that considering the attractor equations (28) and (29) which hold in the near horizon region only, one could not determine  $G(r)$ . Also, it is important to know that  $G(r)$  is regular as  $r \rightarrow 0$ , in contrast to the Reissner-Nordström case ( $a = 0$  limit) where its metric function of  $1 - 2M/r + Q^2/r^2$  diverges as  $r^{-2}$  in that limit. In this sense, the regularity is understood here as the regularity of line element rather than the regularity of spacetime.

In order to find the location  $r = r_{\pm}$  of event horizon from  $G(r) = 0$ , we use the Lambert functions  $W_i(\xi)$  defined by the general formula  $e^{W(\xi)}W(\xi) = \xi$  [12]. Here  $W_0(\xi)$  and  $W_{-1}(\xi)$  have real branches, as is shown in Fig. 1a. Their values at branch point  $\xi = -1/e$  are the same as  $W_0(-1/e) = W_{-1}(-1/e) = -1$ . Here we set  $W_0(1/e) \equiv w_0$  because the Lambert function at  $\xi = 1/e$  plays an important role in finding the location  $r = r_{ext}$  of degenerate horizon for an extremal RBH. For simplicity, let us introduce a reduced radial coordinate  $x = r/M$  and a charge-to-mass ratio  $q = Q/M$  to find the outer  $x_+$  and inner  $x_-$  horizons as

$$x_+ = -\frac{q^2}{W_0(-\frac{q^2 e^{q^2/4}}{4}) - q^2/4}, \quad x_- = -\frac{q^2}{W_{-1}(-\frac{q^2 e^{q^2/4}}{4}) - q^2/4}. \quad (13)$$

Especially for  $q = q_{ext} = 2\sqrt{w_0}$  when  $(q_{ext}^2/4)e^{q_{ext}^2/4} = 1/e = w_0 e^{w_0}$ , the two horizons  $r_+$  and  $r_-$  merge into a degenerate event horizon<sup>4</sup> at

$$x_{ext} = \frac{4q_{ext}^2}{4 + q_{ext}^2} = \frac{4w_0}{1 + w_0}. \quad (14)$$

This is shown in Fig. 1b. Alternatively, in addition to  $G(r) = 0$ , requiring a further condition

$$G'(r) = 0, \quad (15)$$

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<sup>4</sup>For the Reissner-Nordström black hole ( $a = 0$  limit), we have the outer  $r_+$  and inner  $r_-$  horizon at  $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ . Further, its degenerate event horizon appears at  $r_{ext} = M = Q$ . In terms of  $x_{\pm} = r_{\pm}/M$  and  $q = Q/M$ , we have  $x_{\pm} = 1 \pm \sqrt{1 - q^2}$ . In the case of extremal black hole ( $q_{ext}^2 = 1$ ), one has  $x_{\pm} = x_{ext} = 1$ . Its entropy is given by  $S_{BH}^{RN} = \pi M^2 x_{ext}^2 = \pi Q^2$ .

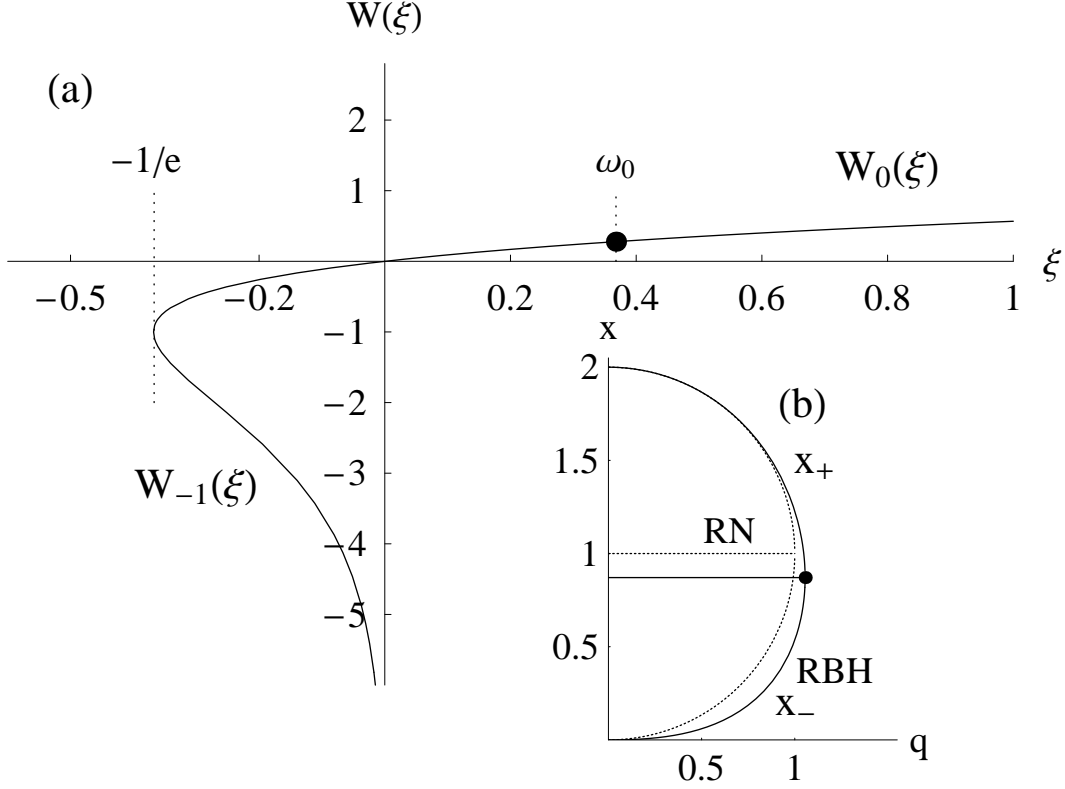


Figure 1: (a) Two real branches of the Lambert function  $W_0(\xi)$  (upper curve) and  $W_{-1}(\xi)$  (lower curve) are depicted for solution to the RBH. The degenerate event horizon at  $(q_{ext}, x_{ext})$  corresponds to the branch point of the Lambert function at  $\xi = -1/e$ . (b) Graphs for horizons  $x_+$  and  $x_-$  as the solution to  $G(r) = 0$ . The solid line denotes the magnetically charged RBH, while the dotted curve is for the RN black hole ( $a = 0$  limit). A dot (•) represents the position of extremal RBH which satisfies  $G'(r) = 0$  further.

one arrives at the same location of degenerate horizon as in Eq. (14). Here ' denotes the derivative with respect to  $r$ . For  $q > q_{ext}$ , there is no horizon. In Fig. 1b we have shown that the solid line denotes the magnetically charged RBH: the upper curve describes the outer horizon  $x_+$  while the lower curve the inner horizon  $x_-$ , separated by the real branches of the Lambert function. The degenerate event horizon appears at  $(q_{ext} = 1.056, x_{ext} = 0.871)$ . On the other hand, the dotted curve is for the Reissner-Norström black hole where the upper curve describes the outer horizon, while the lower

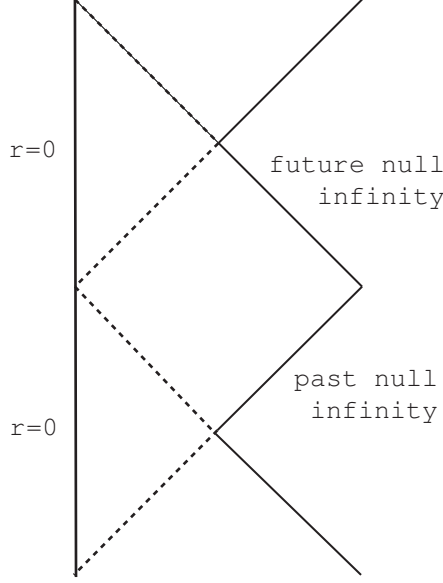


Figure 2: The Penrose diagram of extremal RBH. The left (oblique) lines denote  $r = 0$  ( $r = \infty$ ), while the dotted lines represent the degenerate horizon  $r = r_{ext}$ . This diagram is identical to the extremal RN black hole except replacing the wave line at  $r = 0$  by the solid line.

curve the inner horizon. These are coalesced into the extremal point at  $(q_{ext}, x_{ext}) = (1, 1)$ , which is different from that of the nonlinear Maxwell case of the magnetically charged RBH.

The causal structure of the RBH is similar to that of the RN black hole, with the internal singularities replaced by regular centers [5]. As is shown in Fig. 2, the Penrose diagram of the extremal RBH is identical to that of the extremal RN black hole except replacing the wave line at  $r = 0$  by the solid line [26].

We are in a position to investigate the near horizon geometry of the degenerate horizon  $G(r) \simeq D(r - r_{ext})^2$  defined by  $G'(r_{ext}) = 0$  and  $G''(r_{ext}) = 2D$ . For this purpose, one could introduce new coordinates  $r = r_{ext} + \varepsilon/(Dy)$  and  $\tilde{t} = t/\varepsilon$  with

$$D = \frac{(1 + \omega_o)^3}{32M^2\omega_o^2}. \quad (16)$$

Expanding the function  $G(r)$  in terms of  $\varepsilon$ , retaining quadratic terms and subsequently taking the limit of  $\varepsilon \rightarrow 0$ , the line element [12] becomes

$$ds_{NH}^2 \simeq \frac{1}{Dy^2} (-dt^2 + dy^2) + r_{ext}^2 d\Omega_2^2. \quad (17)$$

Moreover, using the Poincaré coordinate  $y = 1/u$ , one could rewrite the above line element

as the standard form of  $AdS_2 \times S^2$

$$ds_{NH}^2 \simeq \frac{1}{D} \left( -u^2 dt^2 + \frac{1}{u^2} du^2 \right) + r_{ext}^2 d\Omega_2^2. \quad (18)$$

In the case of the Reissner-Nordström black hole, we have the Bertotti-Robinson geometry with  $1/D = r_{ext}^2 = Q^2$ .

For our purpose, let us define the Bekenstein-Hawking entropy for the magnetically charged extremal RBH

$$S_{BH} = \pi r_{ext}^2 = \pi M^2 x_{ext}^2 = \pi Q_{ext}^2 \left[ \frac{4q_{ext}}{4 + q_{ext}^2} \right]^2 \quad (19)$$

with  $Q_{ext} = Mq_{ext}$ . On the other hand, it is a nontrivial task to find the higher curvature corrections to the Bekenstein-Hawking entropy in Eq.(19) when considering together Einstein gravity-nonlinear electromagnetics with the higher curvature terms [13].

### 3 Entropy of extremal RBH

Since the magnetically charged extremal RBH is an interesting object whose near horizon geometry is given by topology  $AdS_2 \times S^2$  and whose action is already known, we attempt to obtain the black hole entropy in Eq.(19) through the entropy functional approach. According to Sen's entropy function approach, we consider an extremal black hole solution whose near horizon geometry is given by  $AdS_2 \times S^2$  with the magnetically charged configuration

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_2^2, \quad (20)$$

$$F_{\theta\phi} = Q \sin \theta, \quad (21)$$

where  $v_i (i = 1, 2)$  are constants to be determined. For this background, the nonvanishing components of the Riemann tensor are

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -v_1^{-1} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), & \alpha, \beta, \gamma, \delta &= r, t, \\ R_{mnpq} &= v_2^{-1} (g_{mp} g_{nq} - g_{mq} g_{np}), & m, n, p, q &= \theta, \phi, \end{aligned} \quad (22)$$

which are related to  $AdS_2$  and  $S^2$  sectors, respectively. Let us denote by  $f(v_i, Q)$  the Lagrangian density (1) evaluated for the near horizon geometry (20) and integrated over the angular coordinates [27]:

$$f(v_i, Q) = \frac{1}{16\pi} \int d\theta d\phi \sqrt{-g} [R - \mathcal{L}(B)]. \quad (23)$$



Since  $R = -\frac{2}{v_1} + \frac{2}{v_2}$  and  $B = \frac{2Q^2}{v_2^2}$ , we obtain

$$f(v_i, Q) = \frac{1}{2}v_1v_2 \left[ -\frac{1}{v_1} + \frac{1}{v_2} - \frac{1}{2}\mathcal{L}(v_2, Q) \right]. \quad (24)$$

Here

$$\mathcal{L}(v_2, Q) = \frac{2Q^2}{v_2^2} \cosh^{-2} \left( \frac{Q^2}{2\alpha\sqrt{v_2}} \right), \quad (25)$$

which is a nonlinear function of  $v_2$ . Further, we choose the free parameter  $a = Q^{3/2}/2\alpha$ . Then, one could obtain the values of  $v_i^e$  at the degenerate horizon by extremizing  $f$ :

$$\frac{\partial f}{\partial v_i} = 0. \quad (26)$$

On the other hand, the non-trivial components of the gauge field equation and the Bianchi identities are already given in Eqs. (3) and (4), which are automatically satisfied by the background (20) and (21). It follows that the constant  $Q$  appearing in (21) corresponds to a magnetic charge of the black hole. For fixed  $Q$ , Eq. (26) provides a set of equations, which are equal in number to the number of unknowns  $v_i$ . Hereafter we choose the free parameter  $a = Q^{3/2}/2M$  ( $\alpha = M$ ) to meet the condition that the near horizon geometry of Eq.(20) reflects that of the magnetically charged extremal RBH. For the magnetically charged extremal RBH, the entropy function is given by

$$\mathcal{F}(v_i, Q) = -2\pi f(v_i, Q). \quad (27)$$

In this case, the extremal values  $v_i^e$  may be determined by extremizing the function  $\mathcal{F}(v_i, Q)$  with respect to  $v_i$ :

$$\frac{\partial \mathcal{F}}{\partial v_1} = 0 \rightarrow \frac{v_2}{2} \mathcal{L}(v_2, Q) = 1 \text{ with } \mathcal{L}(v_2, Q) = \frac{2Q^2}{v_2^2} \cosh^{-2} \left( \frac{Q^2}{2M\sqrt{v_2}} \right), \quad (28)$$

$$\frac{\partial \mathcal{F}}{\partial v_2} = 0 \rightarrow \frac{1}{v_1} = \frac{Q^2}{v_2^2} \cosh^{-2} \left[ \frac{Q^2}{2M\sqrt{v_2}} \right] - \frac{Q^2}{v_2} \frac{\partial}{\partial v_2} \left( \cosh^{-2} \left[ \frac{Q^2}{2M\sqrt{v_2}} \right] \right) \quad (29)$$

which are two attractor equations. Using the above relations, the entropy function at the extremum is given by

$$\mathcal{F}(v_2^e, Q) = \pi v_2^e. \quad (30)$$

In order to find the proper extremal value of  $v_2^e$ , we introduce  $Q = \tilde{q}M$ ,  $v_2^e = M^2\tilde{x}^2$  and  $v_1^e = M^2\tilde{v}_1$ . Then Eqs. (28) and (29) with Eq. (25) can be rewritten as

$$\frac{\tilde{x}^2}{\tilde{q}^2} = \cosh^{-2} \left( \frac{\tilde{q}^2}{2\tilde{x}} \right), \quad (31)$$

$$\frac{1}{\tilde{v}_1} = \frac{\tilde{q}^2}{\tilde{x}^4} \cosh^{-2}(\tilde{q}^2/2\tilde{x}) - \frac{\tilde{q}^4}{2\tilde{x}^5} \frac{\sinh(\tilde{q}^2/2\tilde{x})}{\cosh^3(\tilde{q}^2/2\tilde{x})}, \quad (32)$$

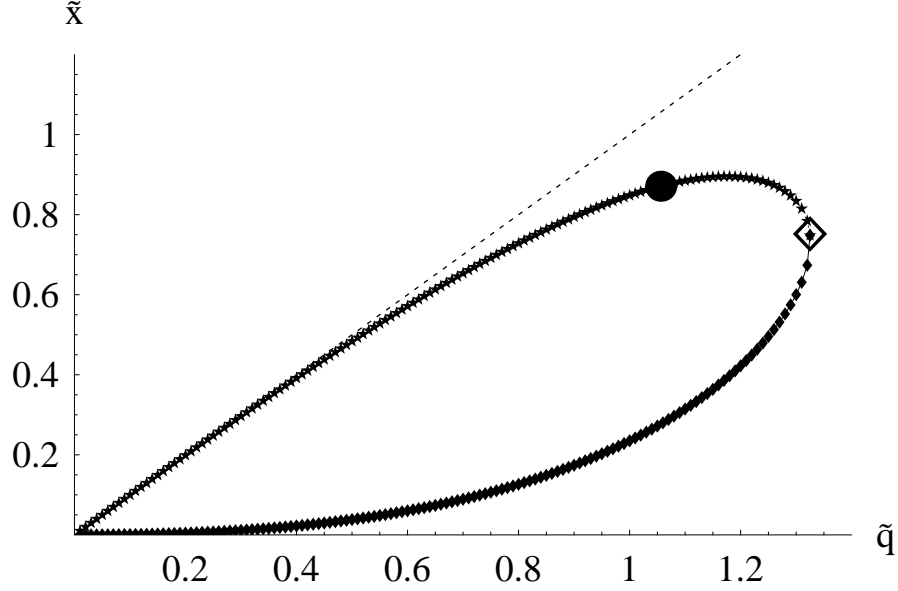


Figure 3: Plot of curvature radius  $\tilde{x}$  of  $S^2$  versus parameter  $\tilde{q}$ . The solid curve with the upper and lower branches denotes the solution space to Eq. (31), while the dotted line represents  $\tilde{x} = \tilde{q}$  for the extremal RN black hole with  $v_2^e = Q^2$ . A dot (•) represents the extremal black hole, whose conditions are given by both  $G(r) = 0$  and  $G'(r) = 0$ . Diamond(◊) denotes the point of  $(\tilde{q}_c, \tilde{x}_c) = (1.325, 0.735)$  at which the upper and lower branches merge.

where we use  $\tilde{x}$  and  $\tilde{q}$  to distinguish  $x$  and  $q$  for the full equations. Note that these equations are identical to those in Ref. [12] derived from the near horizon geometry of an extremal RBH. This means that the entropy function approach is equivalent to solving the Einstein equation on the  $AdS_2 \times S^2$  background, but not the full equations.

Since the above coupled equations are nonlinear equations, we could not solve them analytically. Instead, let us numerically solve the nonlinear equation (31) whose solutions are depicted in Fig. 3. It seems that there are two branches: the upper and lower ones which merge at  $(\tilde{q}_c, \tilde{x}_c) = (1.325, 0.735)$ . Note that the magnetically charged extremal RBH corresponds to the point  $(\tilde{q}_{ext}, \tilde{x}_{ext}) = (1.056, 0.871)$ . However, there is no way to fix this point although the solution space comprises such a point. Hence it seems that the entropy function approach could not explicitly determine the position of  $v_1^e = 1/D$  and  $v_2^e = r_{ext}^2 = M^2 x_{ext}^2$  of the extremal RBH. We note the case of  $G(r) = 0, G'(r) = 0 \rightarrow r = r_{ext}, Q = Q_{ext}$ , which implies  $\frac{v_2^e}{2} \mathcal{L}(v_2^e, Q_{ext}) = 1$  as dot (•) in Fig. 3. On the other hand, the case of  $\frac{v_2}{2} \mathcal{L}(v_2, Q) = 1$  does not lead to the extremal point.

As a result, the entropy function does not lead to the Bekenstein-Hawking entropy

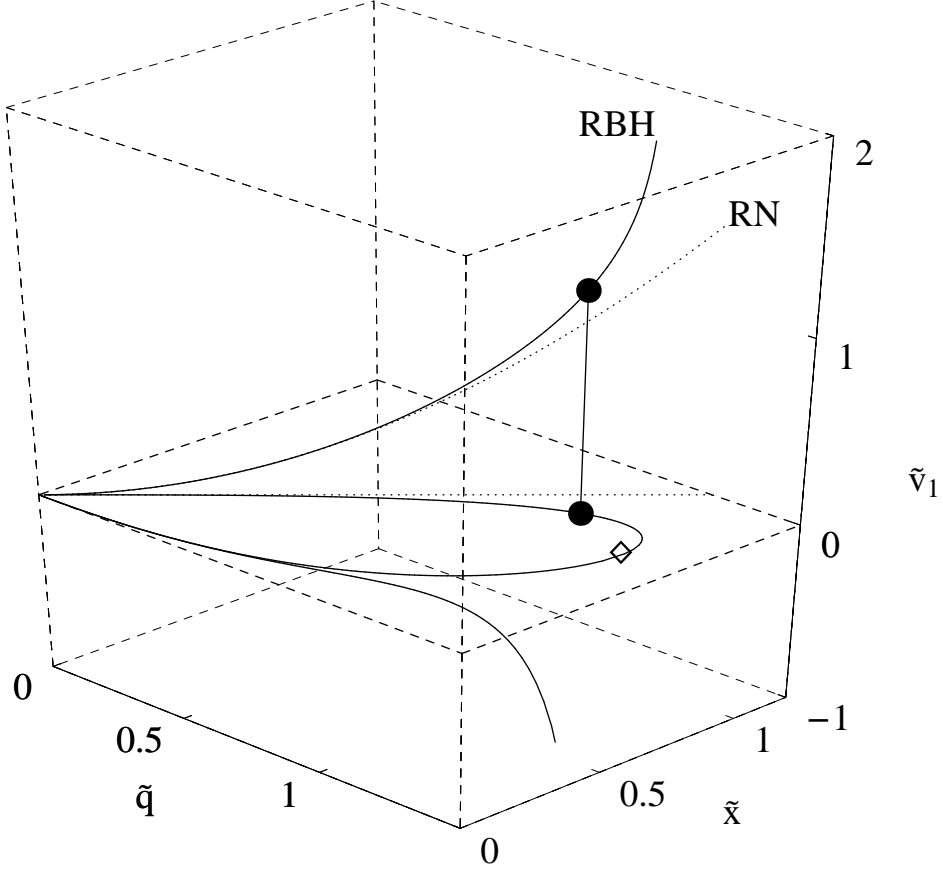


Figure 4: Figure of  $\tilde{v}_1$  as a function of  $\tilde{q}$  and  $\tilde{x}$ . Solid curve  $\tilde{v}_1$ , which is a monotonically increasing function of  $\tilde{q}$  and  $\tilde{x}$  for the upper branch, denotes the solution space to Eq. (32) (attractor equation). The lower branch takes negative values and thus it is ruled out from the solution space. Dotted curve shows that of extremal RN black hole. A dot (•) represents the extremal RBH.

(19) for the case of the magnetically charged extremal RBH as follows:

$$\mathcal{F} = \pi v_2^e = \pi M^2 \tilde{x}^2 \neq S_{BH} = \pi M^2 x_{ext}^2. \quad (33)$$

Therefore, we do not need to consider the higher curvature corrections because the entropy function approach does not work even at the level of  $R$ -gravity.

At this stage, it seems appropriate to comment on the case of the singular Reissner-Nordström black hole on the  $AdS_2 \times S^2$  background ( $a = 0$  limit). In this case, we have

the entropy function as

$$\mathcal{F}^{RN}(v_i, Q) = \pi \left[ v_2 - v_1 + Q^2 \frac{v_1}{v_2} \right]. \quad (34)$$

Considering the extremizing process of  $\partial \mathcal{F}^{RN} / \partial v_i = 0$ , we find  $v_2^e = Q^2 = v_1^e$ , which determines the near horizon geometry of the extremal Reissner-Nordström black hole completely. Then, we obtain the entropy function

$$\mathcal{F}^{RN} = \pi v_2^e = \pi Q^2 = S_{BH}^{RN}, \quad (35)$$

which shows that the entropy function approach exactly reproduces the Bekenstein-Hawking entropy, in contrast to the case of the magnetically charged extremal RBH. Note that introducing  $v_2^e = M^2 \tilde{x}^2$ ,  $Q = M \tilde{q}$  and  $v_1^e = M^2 \tilde{v}_1$ , one obtains the relation of  $\tilde{x} = \tilde{q}$  from  $v_2^e = Q^2$ . Furthermore, the straight line in Fig. 3 shows the extremal Reissner-Nordström black hole solution.

In order to find more information from Eq. (32), let us solve it numerically for given  $(\tilde{q}, \tilde{x})$ . The corresponding three dimensional graph is shown in Fig. 4. It shows that for the upper branch,  $\tilde{v}_1$  is a monotonically increasing function of  $\tilde{q}$  and  $\tilde{x}$  and thus the extremal point  $(\tilde{q}_{ext}, \tilde{x}_{ext}, 1/D)$  is nothing special. Since the lower branch takes negative value of  $1/\tilde{v}_1$ , it does not belong to the real solution space. Therefore, we could not find the regular extremal point  $(1.056, 0.871, 1.188)$  with  $M = 1$  even for including the curvature radius  $1/\tilde{v}_1$  of  $AdS_2$ -sector. On the other hand, for the singular case of the Reissner-Nordström black hole, the corresponding relation is given by  $\tilde{v}_1 = (\tilde{q}^2 + \tilde{x}^2)/2$ .

Finally, we would like to mention how to derive the Bekenstein-Hawking entropy of the extremal RBH from the generalized entropy formula based on the Wald's Noether charge formalism [25]. According to this approach, the entropy formula takes the form

$$S_{BH} = \frac{4\pi}{G''(r_{ext})} (qe - F(r_{ext})), \quad (36)$$

where the generalized entropy function  $F$  is given by

$$F(r_{ext}) = \frac{1}{16\pi} \int_{r=r_{ext}} d\theta d\varphi r^2 [R - \mathcal{L}_M(r, Q)] \quad (37)$$

with the curvature scalar and the matter

$$R = -\frac{r^2 G'' + 4r G' + 2G - 2}{r^2},$$

$$\mathcal{L}_M(r, Q) = \frac{2Q^2}{r^4} \cosh^{-2} \left[ \frac{Q^2}{2Mr} \right]. \quad (38)$$

In this approach, one has to know the location  $r = r_{ext}$  of degenerate event horizon (solution to full Einstein equation:  $G(r) = 0, G'(r) = 0$ ). After the integration of angular coordinates, the generalized entropy function leads to

$$F(r_{ext}) = \frac{1}{4} \left[ -r^2 G''(r) + 2 - r^2 \mathcal{L}_M(r, Q) \right] \Big|_{r=r_{ext}} = -\frac{1}{4} G'''(r_{ext}) r_{ext}^2 \quad (39)$$

because of  $\mathcal{L}_M(r_{ext} = Mx_{ext}, Q_{ext} = Mq_{ext}) = 2/r_{ext}^2$ . For a magnetically charged RBH with  $e = 0$ , we have the correct form of entropy from Eq. (36)

$$S_{BH} = -\frac{4\pi}{G''(r_{ext})} F(r_{ext}) = \pi r_{ext}^2. \quad (40)$$

Even though we find the prototype of the Bekenstein-Hawking entropy using the entropy formula based on Wald's Noether charge formalism, there is still no way to explicitly fix the location  $r = r_{ext}$  of degenerate horizon.

## 4 Discussions

We have considered a magnetically charged RBH in the coupled system of the Einstein gravity and nonlinear electrodynamics. The black hole solution is parameterized by the ADM mass and magnetic charge  $(M, Q)$ , while the free parameter  $a$  is adjusted to make the resultant line element regular at the center. Here we have put special emphasis on its extremal configuration because it has the similar near horizon geometry  $AdS_2 \times S^2$  of the extremal Reissner-Nordström black hole ( $a = 0$  limit). However, the near horizon geometry of the magnetically charged extremal RBH (extremal Reissner-Nordström black hole) have different modulus of curvature (the same modulus). Moreover the extremal RBH is regular inside the event horizon, whereas the extremal Reissner-Nordström black hole is singular.

In this work, we have carefully investigated whether the entropy function approach does also work for deriving the entropy of a magnetically charged extremal regular black hole in the Einstein gravity-nonlinear electrodynamics. It turns out that the entropy function approach does not lead to a correct entropy of the Bekenstein-Hawking entropy even at the level of  $R$ -gravity. This contrasts to the case of the extremal Reissner-Nordström black hole in the Einstein-Maxwell theory. This is mainly because the magnetically charged extremal RBH comes from the coupled system of the Einstein gravity and nonlinear electrodynamics with a free parameter  $a \neq 0$ .

It seems that the entropy function approach is sensitive to whether the nature of the central region of the black hole is regular or singular. In order to study this issue

further, one may consider another non-linear term of the Born-Infeld action instead of the nonlinear electrodynamics on the Maxwell-side. It turned out that for a singular black hole with four electric charges, the entropy function approach does not lead to the Bekenstein-Hawking entropy [24]. This means that the Einstein gravity-Born-Infeld theory do not have a nice extremal limit when using the entropy function approach. Hence, we suggest that the nonlinearity on the Maxwell-side makes the entropy function approach useless in deriving the entropy of the extremal black hole.

Furthermore, we mention the attractor mechanism. The entropy function approach did not work because the free parameter is fixed to be  $a = Q^{3/2}/2M$ . This could be explained by the attractor mechanism which states that the near horizon geometry of the extremal black holes depends only on the charges carried by the black hole and not on the other details of the theory [22]. Thus the dynamics on the horizon is decoupled from the rest of the space. The attractor mechanism plays an important role in the entropy function approach. However, this mechanism is unlikely applied to computing the entropy of a magnetically charged extremal regular black hole because the parameter  $a$  depends on both the charge  $Q$  and the asymptotic value  $M$ .

We would like to emphasize our three figures again because these provide the important message to the reader. Fig. 1a and 1b show the outer horizon  $r_+ = Mx_+$  and inner one  $r_- = Mx_-$ , as the solution space to  $G(r) = 0$ . In order to find two horizons, we need to solve the full equation (5) with the boundary conditions at  $r = 0$  (regularity) and  $r = \infty$  (ADM mass). The location of degenerate horizon  $(q_{ext}, x_{ext})$  is determined by requiring the further condition of  $G'(r) = 0$ . Fig. 3 shows the solution space to the attractor equation (28). This equation is not sufficient to determine the location of degenerate horizon, even though the solution space comprises such a degenerate point. Fig. 4 implies that the lower branch in Fig. 3 is meaningless. Consequently, to determine the entropy of an extremal RBH, we need to know the mechanism which translate the full equation to determine  $G(r) = 0$  into the extremal process of attractor equation.

Finally, we note that the failure of the entropy function approach to a magnetically charged extremal RBH is mainly due to the nonlinearity of the matter action (2) with  $a \neq 0$ . Of course, this nonlinear action is needed to preserve the regularity at the origin of coordinate  $r = 0$ . Furthermore, the regular condition of  $a = Q^{3/2}/2M$  requires an asymptotic value of the ADM mass  $M$ , in addition to charge  $Q$ . Considering the  $a = 0$  limit, we find the linear action of the Einstein-Maxwell field, where the entropy function approach works well for obtaining the extremal RN black hole. For the nonextremal RBH, the Bekenstein-Hawking entropy provides  $S_{BH} = \pi r_+^2 = \pi M^2 x_+^2$  as its entropy because the entropy function approach was designed only for finding the entropy of extremal black

holes.

In conclusion, we have explicitly shown that the entropy function approach does not work for a magnetically charged extremal regular black hole, which is obtained from the coupled system of the Einstein gravity and nonlinear electrodynamics.

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## References

- [1] P. H. Ginsparg and M. J. Perry, Nucl. Phys. B **222** (1983) 245;  
O. B. Zaslavsky, Phys. Rev. D **56** (1997) 2188 [Erratum-ibid. D **59** (1999) 069901];  
M. Caldarelli, L. Vanzo, and S. Zerbini, hep-th/0008136;  
O. J. C. Dias and J. P. S. Lemos, Phys. Rev. **D68** (2003) 104010;  
V. Cardoso, O. J. C. Dias, and J. P. S. Lemos, Phys. Rev. **D70** (2004) 024002.
- [2] J. Bardeen, In Proceedings of GR5, Tiflis, U.S.S.R, 1968, p.174;  
J. Bardeen, Phys. Rev. Lett. **46** (1981) 382.
- [3] A. Borde, Phys. Rev. **D 50** (1994) 3692;  
E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. **80** (1998) 5056.
- [4] I. Dymnikova, Gen. Rel. Grav. **24** (1992) 235.
- [5] S. A. Hayward, Phys. Rev. Lett. **96** (2006) 031103.
- [6] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, gr-qc/0702145.
- [7] A. Bonanno and M. Reuter, Phys. Rev. **D 73** (2006) 083005.

- [8] T. G. Rizzo, JHEP **09** (2006) 021;  
P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. **B 632** (2006) 547;  
S. Ansoldi, P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. **B 645** (2007) 261;  
Y. S. Myung, Y.-W. Kim, and Y.-J. Park, JHEP **02** (2007) 012.
- [9] E. Ayon-Beato and A. Garcia, Phys. Lett. **B464** (1999) 25.
- [10] K. A. Bronnikov, Phys. Rev. **D63** (2001) 044005.
- [11] J. Matyjasek, Phys. Rev. **D63** (2001) 084004.
- [12] J. Matyjasek, Phys. Rev. D **70** (2004) 047504;
- [13] W. Berej, J. Matyjasek, D. Tryniecki, and M. Woronowicz, Gen. Rel. Grav. **38** (2006) 885.
- [14] B. Zwiebach, Phys. Lett. B **156** (1985) 315;  
C. G. Callan, I. R. Klebanov, and M. J. Perry, Nucl. Phys. B **278** (1986) 78;  
D. J. Gross and E. Witten, Nucl. Phys. B **277** (1986) 1.
- [15] C. G. Callan, R. C. Myers, and M. J. Perry, Nucl. Phys. B **311** (1989) 673;  
R. C. Myers, gr-qc/9811042.
- [16] B. de Wit, Fortsch. Phys. **54** (2006) 183;  
T. Mohaupt, hep-th/0512048.
- [17] R. M. Wald, Phys. Rev. D **48** (1993) 3427;  
T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49** (1994) 6587;  
V. Iyer and R. M. Wald, Phys. Rev. D **50** (1994) 846;  
T. Jacobson, G. Kang, and R. C. Myers, gr-qc/9502009.
- [18] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lust, T. Mohaupt, and W. A. Sabra,  
Phys. Lett. B **429** (1998) 289;  
A. Sen, JHEP **0505** (2005) 059.
- [19] A. Dabholkar, R. Kallosh, and A. Maloney, JHEP **0412** (2004) 059;  
D. Bak, S. Kim, and S. J. Rey, hep-th/0501014.



- [20] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, Phys. Rev. D **72** (2005) 124021;  
M. Alishahiha and H. Ebrahim, JHEP **0603** (2006) 003;  
R. Kallosh, N. Sivanandam, and M. Soroush, JHEP **0603**(2006) 060;  
A. Sinha and N. V. Suryanarayana, Class. Quant. Grav. **23** (2006) 3305;  
B. Chandrasekhar, S. Parvizi, A. Tavanfar, and H. Yavartanoo, JHEP **0608** (2006) 004.
- [21] D. Astefanesei, K. Goldstein, and S. Mahapatra, hep-th/0611140.
- [22] A. Sen, JHEP **0509**(2005) 038;  
A. Sen, JHEP **0603** (2006) 008.
- [23] B. Sahoo and A. Sen, JHEP **0607** (2006) 008;  
R. G. Cai and D. W. Pang, Phys. Rev. D **74** (2006) 064031;  
R. G. Cai and D. W. Pang, JHEP **0704** (2007) 027;  
S. Hyun, W. Kim, J.J. Oh, and E.J. Son, JHEP **0704** (2007) 057;  
M. R. Garousi and A. Ghodsi, JHEP **0705** (2007) 043;  
M. R. Garousi and A. Ghodsi, arXiv:0705.2149 [hep-th].
- [24] B. Chandrasekhar, Braz. J. Phys. **37** (2007) 349.
- [25] R. G. Cai and L. M. Cao, Phys. Rev. D **76** (2007) 064010 [arXiv:0704.1239 [hep-th]].
- [26] B. Carter, Phys. Lett. **21** (1966) 423.
- [27] A. Sen, JHEP **0507** (2005) 073.